Bayesian Multivariate Density Regression with Coordinate-Wise Predictor Selection

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1 Introduction

We discuss the estimation of the joint density of a multivariate random variable in the presence of **categorical covariates**.

Focusing on density regression rather than modeling summaries, such as expected value or variance, provides a more complete description of the phenomenon of interest.

Goals:
- density estimation of bounded continuous multivariate responses,
- separate variable selection for each dimension of the response.

2 Contributions

We propose a Bayesian approach for conditional multivariate density estimation that

- employs a **Gaussian copula** to model dependence across dimensions,
- employs **mixtures of truncated normals with common atoms** to model marginal distributions,
- models covariate-dependent mixture weights through a **tensor factorization**,
- replaces mode matrices in the tensor factorization with **partitions over the covariate levels**.

3 Notation

Statistical unit i

multivariate response, $\mathbf{x}_i = (x_{1,i}, \dots, x_{d,i})^{\mathrm{T}} \in [A, B]^d$

covariate combination, $\mathbf{c}_i = (c_{1,i}, \dots, c_{p,i})^{\mathrm{T}} \in \{1, \dots, d_1\} \times \cdots \times \{1, \dots, d_p\}$

Tensor factorization for the ℓ^{th} dimension of the response

partition for covariate h, $\mathbf{s}_{\ell,h} = \{s_{\ell,h}^{(1)}, \dots, s_{\ell,h}^{(d_h)}\}^{\mathrm{T}} \in \{1. \dots, d_h\}^{d_h}$

core tensor element in position $\mathbf{m}$, $\boldsymbol{\lambda}_{\ell,\mathbf{m}} = \{\lambda_{\ell,\mathbf{m}}(1), \dots, \lambda_{\ell,\mathbf{m}}(K)\}^{\mathrm{T}} \in \Delta^{K-1}$

mapping of covariate combination $\mathbf{c}_i$, $\mathbf{s}_\ell^{(\mathbf{c}_i)} = \{s_{\ell,1}^{(c_{1,i})}, \dots, s_{\ell,p}^{(c_{p,i})}\}^{\mathrm{T}} \in \{1, \dots, d_1\} \times \cdots \times \{1, \dots, d_p\}$

4 Model specification

Gaussian copula with mixtures of truncated normals with common atoms as marginals

$$f_{\mathbf{x}|\mathbf{c}}(\mathbf{x}_i \mid \mathbf{c}_i) = |\mathbf{R}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mathbf{y}_i^{\mathrm{T}}(\mathbf{R}^{-1} - \mathbf{I}_d)\mathbf{y}_i\right\} \prod_{\ell=1}^{d} \left\{\sum_{k=1}^{K} \lambda_{\ell,\mathbf{s}_\ell^{(\mathbf{c}_i)}}(k) \mathsf{TN}\left(x_{\ell,i}; \mu_k, \sigma_k^2, [A, B]\right)\right\}$$

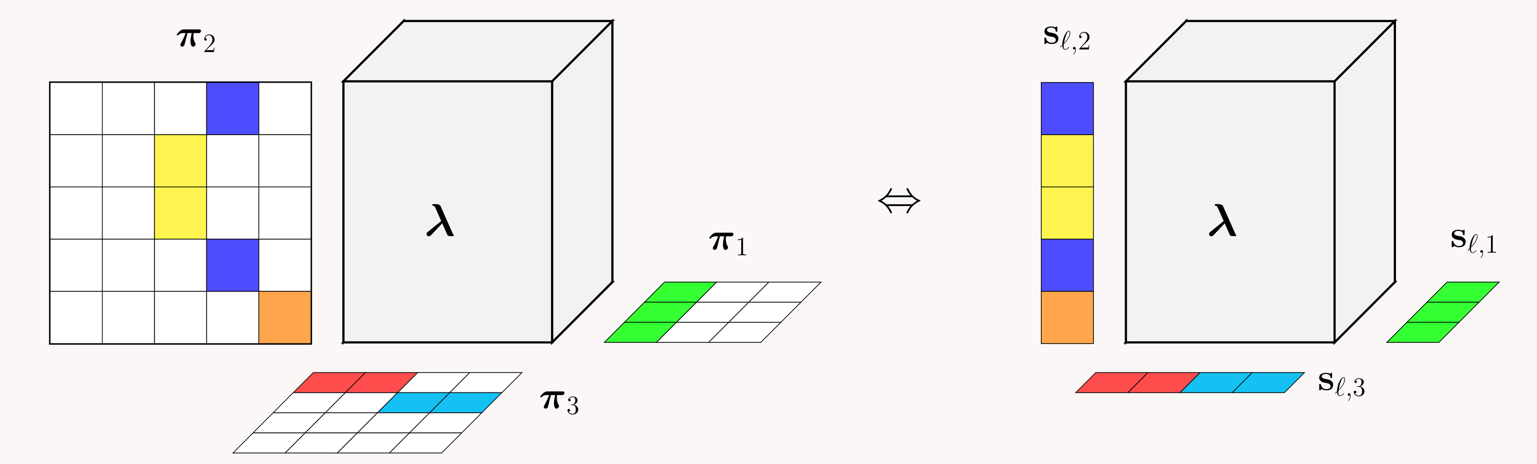
Covariate-dependent weights modeled through a tensor factorization

$$\lambda_{\ell,s_1,\dots,s_p}(k) = \sum_{m_1=1}^{d_1} \cdots \sum_{m_p=1}^{d_p} \lambda_{\ell,m_1,\dots,m_p}(k) \underbrace{\prod_{h=1}^{p} 1(s_h = m_h)}_{\text{selects a single probability}} \qquad \text{with} \quad s_h = s_{\ell,h}^{(c_{h,i})}$$

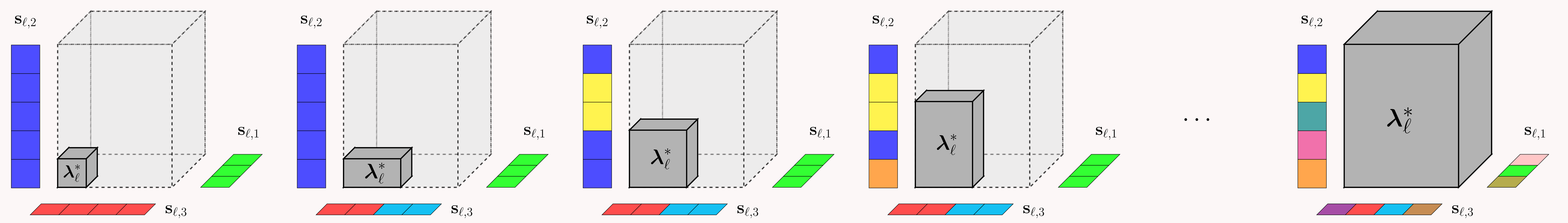
Partitions over covariate levels

$$\{s_{\ell,h}^{(1)}, \dots, s_{\ell,h}^{(d_h)}\} \mid \boldsymbol{\pi}_{\ell,h} \sim \mathsf{Cat}_{d_h}(\boldsymbol{\pi}_{\ell,h}) \times \cdots \times \mathsf{Cat}_{d_h}(\boldsymbol{\pi}_{\ell,h})$$

5 Mode matrices vs. partitions over covariate levels



6 Tensor evolution



7 Density estimation on aggregate level combinations

The model

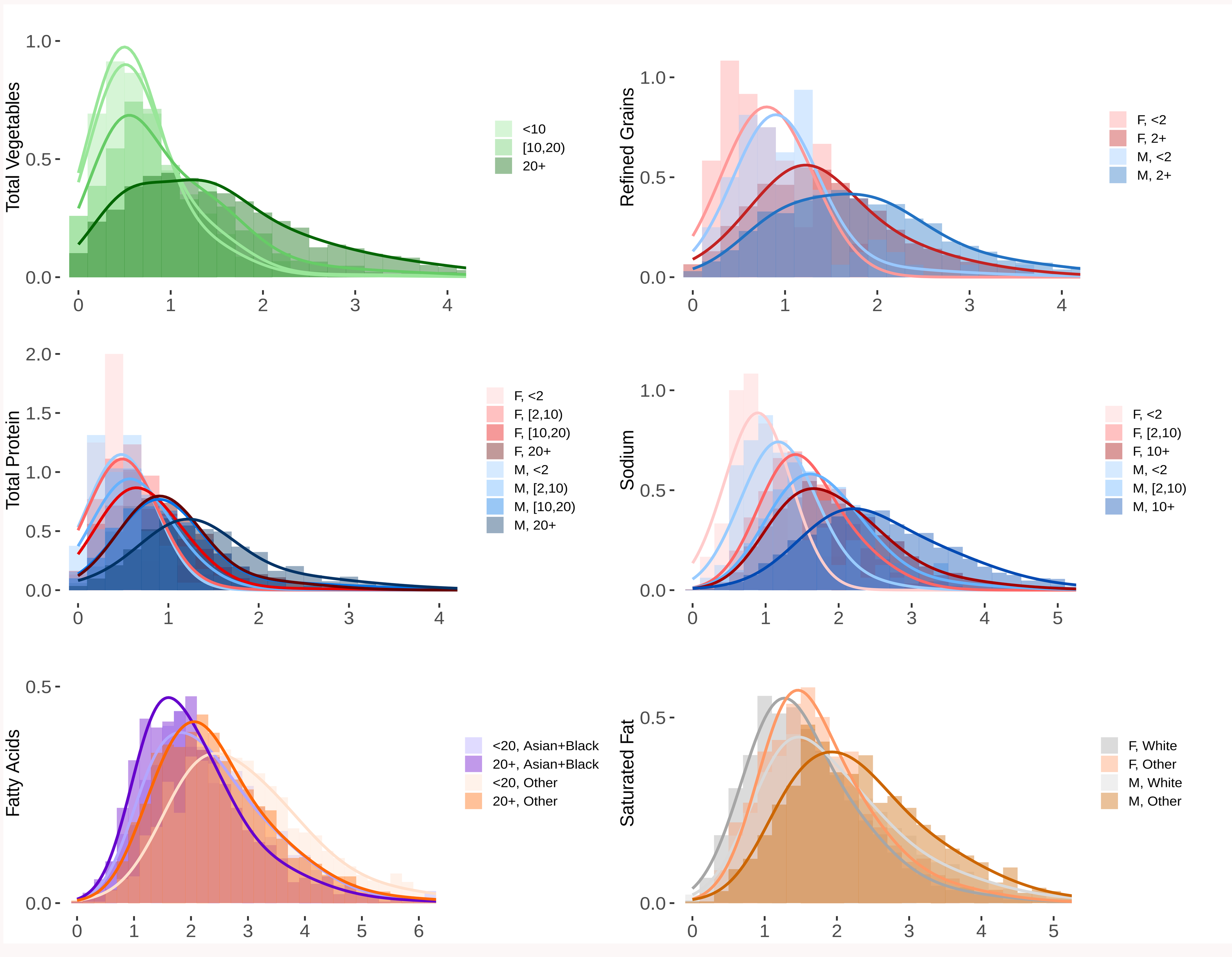
- **aggregates covariate levels** according to the dimension-specific partitions,
- defines **new covariate combinations** based on the aggregate levels,
- assign a **different mixture distribution to each of the new covariate combinations**.

remark: variable selection is a by-product of level aggregation.

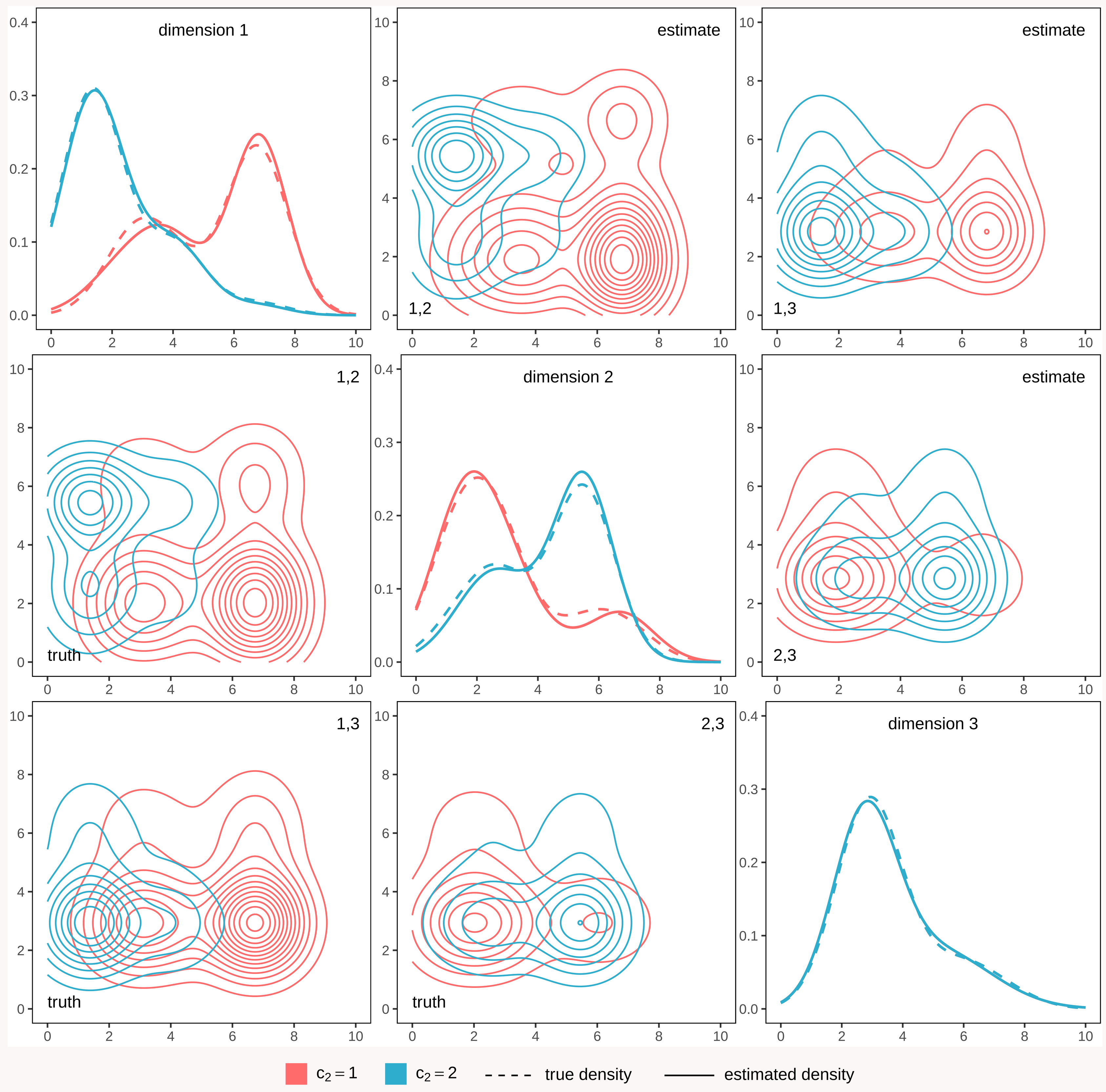
8 NHANES dietary data

Multivariate response: $d = 6$ regularly consumed dietary components

Covariates (n. of levels): sex (2), age (10), race (6), income (17)



9 Simulations



References

Yang, Y. and Dunson, D. B. (2016). Bayesian Conditional Tensor Factorizations for High Dimensional Classification. Journal of the American Statistical Association.

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